'F' - DISTRIBUTION

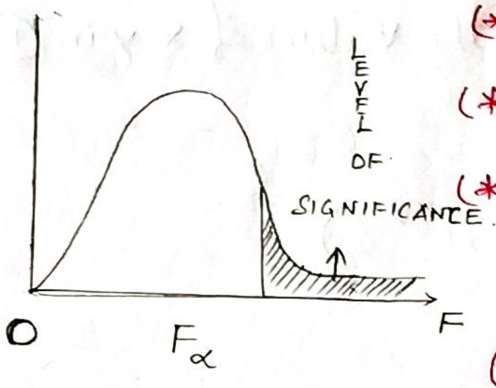
X and Y are the two independent chi - Square Variables. with their degrees of freedom ν_1 and ν_2 respectivly. Then, F- statistical test is defined by,

2M.

$$F = \frac{X/\nu_1}{Y/\nu_2}$$

$\nu \to$ not v it is 'nu'
$\nu_1 = (n_1 - 1)$ sample size
$\nu_2 = (n_2 - 1)$

(i,e) F is defined as the Ratio $(\div)$ of two Independent Chisquar variates (X,Y) divided by their respective degrees of freedom. (ν_1, ν_2)



(*) Skewed at Rightside
(*) Values of F > 0.
(*) Only depends on the Degrees of freedom Values.
(*) Not a bellshaped Curve.

DERIVATION OF SNEDECOR'S F_DISTRIBUTION.

X, Y are the two independent chisquare Variates ν_1, ν_2 are their degrees of freedom respectivly. Then their joint Probability density function is given as,

$$f(x,y) = \left\{ \frac{1}{2^{\nu_1/2}\,\Gamma(\nu_1/2)} \exp(-x/2)\, x^{(\nu_1/2)-1} \right\} \times \left\{ \frac{1}{2^{\nu_2/2}\,\Gamma(\nu_2/2)} \exp(-y/2)\, y^{(\nu_2/2)-1} \right\}$$

This canbe Written as,

$$= \frac{1}{2^{(\nu_1/2 + \nu_2/2)}\,\Gamma(\nu_1/2)\,\Gamma(\nu_2/2)}\, e^{-\frac{x}{2} + y/2} \times x^{(\nu_1/2)-1}\, y^{(\nu_2/2)-1}.$$

$$= \frac{1}{2^{\left(\frac{\nu_1+\nu_2}{2}\right)}\Gamma(\nu_1/2)\,\Gamma\left(\frac{\nu_2}{2}\right)}\, e^{-\left(\frac{x+y}{2}\right)} \times x^{\left(\frac{\nu_1}{2}\right)-1}\, y^{\left(\frac{\nu_2}{2}\right)-1} \longrightarrow ①.$$

$$0 \le (x,y) < \infty.$$

Transformation of Variables are given by,

Consider $F = \frac{X/V_1}{Y/V_2}$

Which can be written $F = \frac{X}{V_1} \times \frac{V_2}{Y}$ substitute $\boxed{Y = u}$ so,

$$F = \frac{X}{V_1} \times \frac{V_2}{u}$$

Finding the Value of $\frac{F}{X} = \frac{1}{V_1} \times \frac{V_2}{u}$.

$$X = \frac{V_1}{1} \times \frac{u}{V_2} F \Rightarrow \boxed{\frac{V_1}{V_2} Fu = X}$$

Jacobian transformation J is given by, Partially differentiating w.r.to u and F,

$$J = \frac{\partial(x,y)}{\partial(F,u)} = \begin{vmatrix} \frac{V_1}{V_2}u & 0 \\ \frac{V_1}{V_2}F & 1 \end{vmatrix}$$

$$= \frac{V_1}{V_2}u - 0 = \boxed{\frac{V_1}{V_2}u}$$

Thus the joint probability density function of the transformed Variables is,

Values of X,Y

(substituting X,Y Values in ①).

$$g(F,u) = \frac{1}{2^{\left(\frac{\nu_1+V_2}{2}\right)}\Gamma\left(\frac{V_1}{2}\right)\Gamma(V_2/2)}\, e^{\left(-\frac{V_1}{V_2}Fu - \frac{u}{2}\right)} \times \left(\frac{V_1}{V_2}Fu\right)^{(V_1/2)-1} u^{\left(\frac{V_2}{2}-1\right)}$$

$$= \frac{(V_1/V_2)^{V_1/2}}{2^{\frac{V_1+\nu_2}{2}}\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{V_2}{2}\right)}\, e^{-\frac{u}{2}\left(1+\frac{V_1}{V_2}F\right)} \times F^{\left(\frac{V_1}{2}\right)-1}\, u^{\left(\frac{V_1+V_2}{2}\right)-1}$$

$$=$$

Integrating w.r. to u over the limit 0 to ∞, then,

$$g_1(F) = \frac{(\nu_1/\nu_2)^{(\nu_1/2)} F^{(\nu_1/2)-1}}{2^{\left(\frac{\nu_1+\nu_2}{2}\right)}\Gamma(\nu_1/2)\Gamma\left(\frac{\nu_2}{2}\right)} \times \left[\int_0^\infty \exp\left\{-\frac{u}{2}\left(1+\frac{\nu_1}{\nu_2}F\right)\right\} u^{\left(\frac{\nu_1+\nu_2}{2}\right)-1} du\right]$$

$$= \frac{(\nu_1/\nu_2)^{(\nu_1/2)} F^{\left(\frac{\nu_1}{2}\right)-1}}{2^{\left(\frac{\nu_1+\nu_2}{2}\right)}\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \times \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\left[\frac{1}{2}\left(1+\frac{\nu_1}{\nu_2}F\right)\right]^{\left(\frac{\nu_1+\nu_2}{2}\right)}}$$ [Using Gamma Integral Formula]

$$g_1(F) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \cdot \frac{F^{\left(\frac{\nu_1}{2}\right)-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}} \qquad 0 \le F < \infty.$$

Which is the required probability function of F-distribution with (ν_1, ν_2) degrees of freedom.

CONSTANTS OF F-DISTRIBUTION.

$$\mu_r' \text{ (about Origin)} = E(F^r) = \int_0^\infty F^r f(F)\, dF.$$

$$\boxed{f(F) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \cdot \frac{F^{(\nu_1/2)-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}}}$$

$$\mu_r' = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \int_0^\infty F^r \frac{F^{(\nu_1/2)-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}} dF.$$

To evaluate the integral, put $y = \frac{\nu_1}{\nu_2}F$

$$dy = \frac{\nu_1}{\nu_2}dF.$$

$$dF = \frac{\nu_1}{\nu_2}dy$$

then we'll get, $\quad \frac{v_1}{v_2} F = y \Rightarrow F = \frac{v_2}{v_1} y$

$$\mu_r' = \frac{\left(\frac{v_1}{v_2}\right)^{v_1/2}}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^\infty F^r \frac{\left(\frac{v_2}{v_1} y\right)^{(v_1/2)-1}}{(1+y)^{\left(\frac{v_1+v_2}{2}\right)}}$$

$$= \frac{\left(\frac{v_1}{v_2}\right)^{v_1/2}}{B\left(\frac{v_1}{v_2}, \frac{v_2}{2}\right)} \int_0^\infty \frac{\left(\frac{v_2}{v_1} y\right)^{r+\left(\frac{v_1}{2}\right)-1}}{(1+y)^{\frac{(v_1+v_2)}{2}}} \left(\frac{v_2}{v_1}\right) dy$$

On Adjusting the Powers of the Constant terms,

$$= \frac{\left(\frac{v_2}{v_1}\right)^r}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^\infty \frac{y^{r+\left(\frac{v_1}{2}\right)-1}}{(1+y)^{v_1/2+r+(v_2/2)-r}} dy$$

$$= \left(\frac{v_2}{v_1}\right)^r \cdot \frac{1}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \cdot B\left(r+\frac{v_1}{2}, \frac{v_2}{2}-r\right)$$

Mode and Points of Inflexion of F-Distribution.

We have,

$$\log f(F) = C + \left[\frac{v_1}{2} - 1\right] \log F - \left(\frac{v_1+v_2}{2}\right) \log\left(1+\frac{v_1}{v_2} F\right)$$

C is the constant independent of F.

Partially differentiating w.r. to f.

$$\frac{\partial}{\partial F}[\log f(F)] = \left(\frac{v_1}{2}-1\right) \cdot \frac{1}{F} - \frac{(v_1+v_2)}{2} \cdot \frac{1}{\left(1+\frac{v_1}{v_2}F\right)} \cdot \frac{v_1}{v_2}.$$

$$f'(F) = \frac{\partial}{\partial F} f(F) = 0.$$

again Partially differentiating w.r. to f equals Zero,

$\Rightarrow$.

$$= \left(\frac{V_1}{2} - 1\right) \cdot \frac{1}{F} - \frac{(V_1+V_2)}{2} \cdot \frac{1}{\left(1+\frac{V_1}{V_2}F\right)} \cdot \frac{V_1}{V_2}$$

$$= \frac{V_1 - 2}{2F} - \frac{V_1+V_2}{2} \cdot \frac{V_1}{V_2 + \frac{V_1 \cancel{V_2}}{\cancel{V_2}} F}$$

$$= \frac{V_1 - 2}{2F} - \frac{V_1+V_2}{2} \cdot \frac{V_1}{V_2 + V_1 F}$$

$$= \frac{V_1 - 2}{2F} - \frac{V_1^2 + V_1 V_2}{2V_2 + 2V_1 F}$$

Taking V_1 from the numerator 2 from the denominator in common from the second Term, we get,

$$\frac{V_1 - 2}{2F} - \frac{V_1(V_1+V_2)}{2(V_2+V_1F)} = 0. \quad \left(\text{Previous paper } \frac{\partial}{\partial F} f(F) = 0\right)$$

On simplification,

$$\frac{V_1 - 2}{2F} - \frac{V_1(V_1+V_2)}{2(V_2+V_1F)} = 0.$$

LCM,

$$\frac{(V_1 - 2)(2V_2 + 2V_1F) - 2FV_1^2 + 2FV_1V_2}{2F \times 2(V_2 + V_1F)} = 0$$

Therefore.

$$2V_1V_2 - \cancel{2V_1^2F} - 4V_2 - 4V_1F - \cancel{2FV_1^2} - 2FV_2V_1 = 0.$$

$$2V_1V_2 - 4V_2 - 4V_1F - 2FV_2V_1 = 0.$$

Taking F terms to the R.H.S.

$$2V_1V_2 - 4V_2 = 4V_1F + 2FV_1V_2.$$

$$2(V_1V_2 - 2V_2) = 2F(2V_1 + V_1V_2)$$

$$\frac{2(V_1V_2 - 2V_2)}{2(2V_1 + V_1V_2)} = F \Rightarrow F = \frac{V_1V_2 - 2V_2}{V_1V_2 + 2V_1}$$

$$F = V_2(V_1 - 2) / V_1(V_2 + 2)$$

$$\therefore F = \frac{v_2(v_1-2)}{v_1(v_2+2)}$$

We can verify that at this point $f''(F) < 0$. Hence

$$\text{mode} = \frac{v_2(v_1-2)}{v_1(v_2+2)}.$$

Example 16:20